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THE GROUP-THEORY ELEMENT OF THE HISTORY OF MATHEMATICS

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FEW mathematical terms suggest such fundamental human cravings as the term group, and few have been more appropriately chosen. Just as human society has led to perplexities which increased with the advance of civilization so the mathematical group-theory has given rise to problems which became more and more difficult with the advances in the development of mathematics. In both cases the primitive stages are comparatively simple and their history throws important light on the later developments.

The history of the mathematical group-theory can be conveniently divided into three periods. The first of these extends from the beginning of mathematical history to about 1770 A. D., and may be called the *implicit period* since the group concept was then employed without being explicitly stated. The second, or *specialization period,* extends from about 1770 to about 1870. During this period the theory of substitution groups was founded as an autonomous science and the usefulness of this theory in the study of algebraic equations was emphasized. The third, or *generalization period,* extends from about 1870 to the present day, and is characterized by increased generalizations by abstraction and the explicit use of groups in each of the large domains of mathematics.

The most fundamental property of the elements of a group in the common restricted sense of this term is that they satisfy the condition that each group contains one and only one element which satisfies the equation

$$xy=z$$

whenever any two of these symbols are replaced by two equal or unequal elements of the group. This condition is evidently satisfied by the members of the number systems of the ancients, including the Babylonians, the Egyptians and the Greeks. On the other hand, the zero of our common modern number system destroys this group property of the entire system. If the zero is excluded the rest of these numbers (real or complex) constitute a group as regards multiplication.

One of the oldest groups of operations in the history of mathematics is the multiplication group whose elements are all the ordinary rational numbers with the exception of zero. This group is used implicitly in the Ahmes papyrus, written about 1700 B. C., since the linear equation and fractions are found frequently in this work. It is an interesting historical fact that for many centuries after this date the ancients seemed to have considered only numbers which are elements of this group, and one is led to inquire to what extent the guiding influence of the group concept was responsible for the late introduction of zero as a number.

It might at first appear that the introduction of zero as a number tended to show that the group concept was not a fundamental guiding principle in the development of arithmetic since it failed to dominate when opposing forces presented themselves. The impression that the group concept did not dominate when zero was introduced is, however, not quite correct, since this number which was rejected by the multiplication group was destined to become the principal element, or the identity, of the addition group. "The stone which the builders rejected, the same is become the head of the corner."

It is an interesting fact that the addition group was made possible by the introduction of zero and by granting full number citizenship to the negative numbers, and the latter was done about the time when the multiplication group was somewhat impaired by the introduction of zero. These important extensions of our number system were completed during the seventeenth century but some steps in this direction had been taken a thousand years earlier especially by the Hindus. In particular, Brahmagupta had already illustrated negative and positive numbers by debts and credits and he observed that a debt subtracted from zero becomes a credit and a credit subtracted from zero becomes a debt, and if one subtracts a debt from a credit or a credit from a debt one obtains the sum.

The long delay in the general introduction of zero as a number seems to show that the ancient peoples held tenaciously to the view that all of the numbers without exception should constitute a group as regards multiplication. It is doubtless true that the group concept was not clearly observed by them and that they could not have given a satisfactory account of the motives which guided them in their efforts for more knowledge about numbers, but the fact that they were guided by this concept seems to be well established.

H. Poincaré pointed out that the same concept guided the ancients in their efforts to secure a knowledge of geometry and he noted that the absence of any direct reference to groups in Euclid's *Elements* was due to the fact that the group notions were among the oldest mathematical notions to be assimilated and hence they did not seem to require explicit mention even at the time of Euclid, notwithstanding the fact that the principal foundation of Euclid's demonstrations is really the group and its properties.[1] In view of the fact that the group notions are so fundamental and elementary that they did not seem to require explicit mention at the time of Euclid it may at first appear strange that during the latter half of the nineteenth century these notions assumed a prominent place in the mathematical literature and that the subject of group-theory began to be regarded as one of the most difficult in the whole range of pure mathematics.

The reason for this change of attitude on the part of the mathematicians is not difficult to discover. As long as only the most general notions of groups were needed the subject was naturally regarded as too elementary to require any special attention. The idea that a set of distinct elements should have the property that any two of them can be combined into one and that this one is also found in the set was illustrated not only by the natural numbers but also by the movements of figures in space, and hence this idea became firmly fixed in the human mind at an early age. It is in accord with the human yearnings for completeness and it is a natural extension of the notion of cyclic changes which were illustrated by the daily and the seasonal apparent movements of the sun.

There is only a short step from this idea of completeness to the idea that a set of distinct elements has the property that when any two of the symbols in the equation $xy=z$ are replaced by distinct or equal elements of the set the resulting linear equation has always one and only one root in the set. If we add to these conditions the condition that the associative law shall be satisfied when the elements of the set are combined we have a complete modern definition of the term group, and it is at once apparent that this definition involves only very fundamental and elementary notions in regard to laws in the world of ideas.

While the general laws of the group are very mild they proved to require exceptions at an early stage in the growth of mathematics. As was noted above the entrance of zero into our number system required some modification of these laws as regards the operation of multiplication. The one law of combination imposed by the group notion was too narrow for the full development of our operations with numbers, where two modes of combination, now known as addition and multiplication, were developed even in prehistoric times. Historically the group concept may therefore be said to embody fundamental laws of combination with which human beings became acquainted in prehistoric times but which had to be violated in certain respects in order to secure the most fruitful mathematical developments.

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[1] H. Poincaré, "On the foundations of geometry," *The Monist,* Vol. 9, (1898) p. 34.

As these laws were not formulated abstractly until about 1870 their early violators were naturally unconscious of the significance of their steps as regards the group concept. In fact, these laws might never have been formulated if it had not been discovered that without violating any of them it was possible to develop a very useful and extensive body of knowledge. The mathematical world discovered this fact by accident and at a comparatively late date. The discovery seems to have been due to the development of a large body of knowledge relating to a special class of groups now known as *substitution groups.*

The fundamental ideas involved in this body of knowledge are also very elementary. About 1770 J. L. Lagrange and others were much interested in the solution of the general equation in one unknown. More than two centuries had then elapsed since several Italian mathematicians had discovered algebraic solutions of the general cubic and the general biquadratic equation. All efforts to obtain a solution of the general quintic had failed and the mathematical world was becoming more and more deeply interested in either making further advances along this line or proving that such advances are impossible.

In the study of the methods which had led to success for the lower degrees it appeared that the number of different formal values which certain unsymmetric rational functions of a number of variables assume when these variables are permuted in every possible manner was of fundamental importance. For instance, the expression $x_1x_2+x_3x_4$ assumes the following three values

$$x_1x_2+x_3x_4,\ x_1x_3+x_2x_4,\ x_1x_4+x_2x_3$$

when the four variables x_1, x_2, x_3, x_4 are permuted in every possible manner. As there are 24 possible permutations of these four variables eight of them transform such a function into itself. These eight permutations constitute an important substitution group known as the octic group.

In general, all the permutations on n variables which transform into itself a certain rational function of these variables constitute a substitution group. Hence the study of such groups seemed important for the purpose of proving the existence or the non-existence of rational functions of a given number of variables which assume a given number of values when these variables are permuted in every possible manner. The concept of a substitution group on n variables is thus seen to be a very elementary one but the study of such groups led to a large body of theorems. Some of these appeared elegant even if they were supposed to apply only to a rather special field.

For about a century mathematicians studied these special groups with only occasional glimpses into their deeper meanings and wider applications. E. Galois, A. L. Cauchy, A. Cayley, and W. R. Hamilton made references to these deeper meanings, especially as regards an abstract theory, but none of these men formulated the abstract laws governing this theory. About 1870 an eminent triumvirate of mathematicians, C. Jordan, S. Lie and F. Klein, began to exhibit the applications of the group concept to new fields. In his "Traité des Substitutions" (1870) and in an article on the groups of movements (1868) C. Jordan made fundamental geometric applications, which were greatly extended by F. Klein. About the same time S. Lie founded a new theory of continuous groups of transformations and made extensive applications of these groups in the theory of differential equations and in other mathematical subjects.

It may be of interest to note that during the first, or implicit period, of the development of our subject, groups involving an infinite number of elements exercised the greatest influence. During the second, or specialization period, the attention was centered on groups of a finite number of elements, while during the third, or generalization period, groups involving an infinite number of elements again moved to the foreground, but groups of finite order continued to receive considerable attention. Two types of groups of infinite order were studied during this period, viz., those in which the transformations were continuous and those in which these transformations were discontinuous.

The fundamental abstract notions involved in group theory are so elementary that they can be easily understood by those who are not professional mathematicians. Hence it is the more interesting that these notions were not explicitly formulated before 1870. In formulating these for the special case when the elements obey the commutative law when they are combined, L. Kronecker expressed himself as follows: "The extremely simple principles upon which the method of Gauss is founded, find applications not only in the place named but also in others, and, indeed, already in the elementary parts of the theory of numbers. This circumstance points to the fact, about which it is easy to convince oneself, that the said principles belong to a more general and more abstract sphere of ideas. Hence it appears appropriate to free their development from all non-essential limitations so that one will be spared the trouble of repeating the same method of reaching a conclusion in the different instances of its use. The advantage of this appears even in the development itself, and the presentation gains at the same time in simplicity, and, by the clear exhibition of the essentials only, also in distinctness when it is given in the most general permissible way."[2]

The student of the history of science may be especially interested in the fact that the formulation of a definition of an abstract group came so late in the development of this subject. For a full century mathematicians were dealing with special substitution groups before making a serious effort to develop an abstract theory embodying the fundamental principles of these groups as a special case. It was not until such an abstract theory was being developed that mathematicians began to see that the group concept had been a dominant factor in some of the most important early mathematical work and hence it became an important means not only for suggesting further advances but also for securing an insight into the large body of earlier mathematical developments.

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[2] L. Kronecker, Berlin *Monatsberichte,* 1831, p. 882.

A few statements found in well-known textbooks may serve to illustrate the attitude of leading mathematicians at the beginning of the present century as regards the theory of groups. In the preface of his "Géometrie," 1905, E. Borel says:

> The new foundation (of elementary geometry) has been laid in the nineteenth century by the works of leading mathematicians. It consists of the recognition that elementary geometry is equivalent to the investigation of the group of movements. Such a view is in accord with the characteristic tendency of modern scientists to replace static investigations of the phenomena by dynamic; or, to speak in more general terms, the thought of development penetrates more and more our observations.

In his "Lehrbuch der Algebra" (kleine Ausgabe), 1912, page 180, H Weber notes that:

> There are chiefly two large general concepts which dominate modern algebra. The existence and importance of these concepts could be observed only after algebra was completed to a certain extent, and had become the property of the mathematicians. Only then could be observed the combining and guiding principles. These are the concepts of groups and domains (koerper) which we now proceed to explain. The more general of these is the concept of group.

In his "Berührungstransformationen," 1914, page 11, H. Liebmann makes the following statement:

> The rules and concept development of group theory may be compared with the organizing laws of nature according to which crystals arise. If it is allowed to continue the figure of speech it may be added that the remaining mother liquor is a rich fostering soil on which luxuriant organized life unfolds itself.

These quotations may suffice to indicate in a general way to what extent group-theory influenced the trend of mathematical progress since the beginning of the third period of its development. The infinite number of finite groups, each of which exhibits special laws of operations, which had been discovered during the second period of the development of this subject, showed that this theory can never be completely mastered in its details. There are, however, large categories of groups which have many properties in common and whose common operational laws throw light on other mathematical developments.

Comparatively little progress has been made in the study of those abstract properties which all groups have in common, yet it is just these common properties which were popularized by the mathematical literature of the last quarter of the nineteenth century. While they are so simple that the ancients did not consider it necessary to mention them explicitly it was found that they furnish a point of view which offers many advantages. For instance, few mathematical terms are more useful than the term equivalent, and one of the services which group-theory has rendered is to give this term a flexible yet perfectly definite meaning by noting that the equivalence of two objects implies that one can be transformed into the other by the operations of a certain group.

Hence the term equivalent is relative to the group under consideration. For instance, in Euclidean geometry two figures are equivalent if they can be made to coincide by operations of the group composed of displacements and symmetries. The distance between any two points is an absolute invariant under this group. On the other hand, in elementary geometry two figures are equivalent when they can be transformed into each other by the operators of the group composed of the similarity transformations which includes the preceding group as an invariant subgroup. In elementary geometry all circles are equivalent, and all squares are equivalent, but this is not true in Euclidean geometry.

Euclid's "Elements" could have been enriched not only by the explicit use of groups of infinite order but also by the introduction of groups of finite order. In particular, the five regular solids which play an importane roôle in Greek mathematics and in Greek philosophy represent three interesting groups of finite order. In the words of E. Picard:

> A regular polyhedron, say an icosahedron, is on the one hand the solid that all the world knows; it is also, for the analyst, a group of finite order, corresponding to the divers ways of making the polyhedron coincide with itself. The investigation of all the types of groups of motion of finite order interests not only the geometers, but also the crystallographers; it goes back essentially to the study of groups of ternary linear substitutions of determinant unity, and leads to the thirty-two systems of symmetry of the crystallography for the particular complex.

While it seems impossible to establish the reasons why Euclid did not make explicit use of groups of finite and of infinite order in his "Elements," the fact that Aristotle frequently expressed the view that mathematics has to do with the *immovable* objects except such as relate to astronomy, is suggestive. While movements were used to illustrate the demonstrations of theorems the Greek philosophers seemed to hold the view that geometry itself was essentially a static subject. It is difficult to overestimate the great influence which this view had on the later history of mathematics.

If Euclid had emphasized in his "Elements" the dynamic rather than the static elements of mathematics it is likely that his work would have exerted a more vigorous influence. The cube of Euclid, for instance, is of great interest but it is not so inspiring as the cube composed of the twenty-four movements of space which leave Euclid's cube invariant. These movements affect all space and convey big and farreaching notions. Moreover, they suggest many questions as regards subgroups and abstract laws of operation. In particular, this group of order 24 is completely defined by the fact that it contains two operators of orders 2 and 3 respectively whose product is of order 4.

While a group-theory of the third century B. C. is conceivable it could not have been the group-theory of the nineteenth century, since the latter century had a much richer mathematical heritage. The rapid strides of group-theory during the last century were largely due to the utilization of old results as is always the case in generalizations by abstraction. The soil had been prepared by the labors of earlier centuries and it was only necessary to sow on it the new seed to secure the bountiful harvest with which the labors of many workers in this field were rewarded, especially during the last decades of the ninteenth century.

When group-theory appeared explicitly it naturally took a form which was in accord with the spirit of the times. Substitution groups constitute a type of combinatory analysis and arose about the time when the Combinatorial School flourished in Germany under the leadership of C. F. Hindenburg (1741-1808). Abstract group-theory is a type of postulational mathematics and its early development during the middle of the preceding century was in the van of the postulational activity which was so prominent during the second half of the nineteenth century. Continuous and geometric group-theory are mainly applied group-theory and their rapid development during the latter quarter of the preceding century is in accord with the spirit of this age when the fear of mathematical isolation through overspecialization tended to make the study of applications especially popular.